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Transverse Shear Effects on Buckling and Postbuckling of Laminated and Delaminated Plates

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The one-dimensional buckling and postbuckling solutions of unsymmetric laminates with clamped edges subjected to constant membrane strain loads are obtained based on the large deflection shear deformation theory. Closed-form expressions of the critical buckling load and postbuckling deflection have been derived. It is found that both effects of bending-extension coupling and transverse shear deformation will reduce the equivalent bending rigidity and buckling load and increase the postbuckling deformation. These solutions are used to study the behavior of a thin-film strip delamination in a base laminate. The analytic form of energy release rate is obtained by a variational energy principle that shows that the slope of the energy release rate with load parameter will be decreased due to effects of both bending-extension coupling and transverse shear deformation.

Nomenclature

A_1, A_2, A_3 = kinematic constants
A_{ij} = extensional stiffnesses, $i,j = 1,2,6$
A_{ij} = interlaminar stiffnesses, $i,j = 4,5$
a = half length of laminate
B_{ij} = bending-extension coupling stiffnesses, $i,j = 1,2,6$
C_{ij} = anisotropic stiffnesses, $i,j = 4,5$
D = equivalent bending rigidity based on shear deformation theory
D_c = equivalent bending rigidity based on classical lamination theory
D_{ij} = bending stiffnesses, $i,j = 1,2,6$
E_f = fracture energy
E_1, E_2 = ply Young's moduli
G = energy release rate
G^* = specific fracture energy
G_{12}, G_{13}, G_{23} = ply shear moduli
$[H]$ = 5×5 matrix defined in Eq. (7)
$[H_{ij}]$ = submatrix of $[H]$
h = thickness of the laminate
k = shear correction factor
L = length of laminate
M_x, M_y, M_{xy} = bending and twisting moment resultants
N_x, N_y, N_{xy} = in-plane force resultants
P = compressive force
P_{cr} = critical buckling load
Q_x, Q_y = transverse shear force resultants
S = in-plane shear force resultant
U_p = strain energy in postbuckling state
U_0 = strain energy in reference state
w = transverse deflection
α, ξ, ζ, η = kinematic constants
γ_{xz}, γ_{yz} = transverse shearing strains in xz and yz planes
Δ = total axial shortening
ϵ^* = equivalent base-laminate strain
ϵ_x^0, ϵ_y^0, γ_{xy}^0 = middle surface strains
$\epsilon_0, \beta_0, \gamma_0$ = membrane strain loads
θ = fiber orientation
κ_x, κ_y, κ_{xy} = middle surface curvatures
ν_{12} = major Poisson's ratio
Π = total potential energy of delaminated layer
σ_x, σ_y, τ_{xy}, τ_{xz}, τ_{yz} = stress components in postbuckling state
σ_{x0}, σ_{y0}, τ_{xy0} = stress components in base laminate
ψ_x, ψ_y = rotations along the x and y directions

Introduction

THE analysis of delamination buckling and growth in composite laminates is an active research topic recently. Both analytical methods[1-7] and numerical methods[8-11] such as the finite element method have been adopted to solve this problem. In many analytical studies, it is assumed that the laminate is a homogeneous isotropic or orthotropic plate.[1,3,6,7] Although most composite laminates in real applications are symmetric, unsymmetric laminates or sublaminates may be formed if delamination occurs. Under this circumstance, the bending-extension coupling effect becomes important and cannot be neglected. Recently, delamination buckling and growth of an unsymmetric laminated structure have been studied by Yin[4,5] based on large deflection classical lamination theory (CLT). In his studies, the bending-extension coupling effect on buckling load, postbuckling deformation, and energy release rate was analyzed. The energy release rate was derived by a path-independent J integral.

The effects of transverse shearing stresses are important for laminated plates because in composite fiber reinforced materials the interlaminar shear moduli are usually much smaller than the in-plane Young's moduli. Although these effects on delamination buckling, postbuckling, and growth in a specially orthotropic rectangular plate under axial loading have been studied in detail previously,[7] they have not been considered in the analytical studies for unsymmetric laminated structures.

In this paper, a one-dimensional thin-film delamination model with unsymmetric laminated structure is developed. Large deflection shear deformation theory (SDT) that includes effects of both bending-extension coupling and transverse shear deformation is used in the formulation. All of the three in-plane loadings—longitudinal membrane strain, transverse membrane strain, and in-plane shearing strain—are applied on the structures. At first, one-dimensional buckling and post-

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buckling behaviors of general unsymmetric laminates are studied. Closed-form expressions of the critical buckling load and postbuckling deflection are obtained. Based on the shear deformation theory, the rotation along the y direction is not trivially zero in order to satisfy the equations of equilibrium, which is different from the case of cylindrical buckling studied previously.[5] It is shown that although the sinusodial terms in extensional and shearing membrane strains are in phase with out-of-plane deflection, the sinusodial terms in rotations along the x and y directions are out of phase with out-of-plane deflection. Among the six elements of the bending-extension coupling matrix $[B_{ij}]$, three terms (B_{11}, B_{16}, B_{66}) have effect on the one-dimensional buckling deformation. The buckling and postbuckling solutions obtained are then taken to analyze the thin-film strip delamination in a thick base laminate. A general strip delamination that takes place at arbitrary depth through the thickness in unsymmetric cross-ply laminates based on shear deformation theory will be studied in a separate paper later.

In the present work, the formula of energy release rate is obtained from the variational energy principle. The same result can also be obtained by a path-independent J integral method. However, with the variational energy principle, the energy release rate can be expressed analytically in terms of in-plane strains, curvatures, resultants of laminate forces, and moments of the delaminated layer and base laminate. It is known that the energy release rate and the stress intensity factor are closely related to the asymptotic behavior of the interlaminar stresses near the crack tip. However, the detailed behavior of the interlaminar stresses near the delamination front are practically irrelevant to the actual evaluation of the J integral for the energy release rate since the path of integration can be moved far from the crack tip. Because the accuracy of the in-plane stresses along the path of integration determines the accuracy of the J integral, a better evaluation for the energy release rate is obtained if the transverse shear effect is taken into account.

One-Dimensional Buckling and Postbuckling Analysis of General Unsymmetric Laminates

Consider a general laminated plate with clamped edges subjected to in-plane membrane loadings. Let ϵ_x^0, ϵ_y^0, and γ_{xy}^0 be longitudinal membrane strain, transverse membrane strain, and in-plane shearing strain of the middle surface, respectively; and κ_x, κ_y, and κ_{xy} the associated curvatures. When these in-plane membrane loadings are sufficiently large, the laminate buckles and deflects from the plane of the laminate. Assume the postbuckling deformation of the laminate to be the following forms:

$$\psi_x = A_1 \sin\frac{\pi x}{a}$$

$$\psi_y = A_2 \sin\frac{\pi x}{a}$$

$$w = A_3\left(1+\cos\frac{\pi x}{a}\right)$$

$$\kappa_x = \psi_{x,x} = \frac{\pi}{a} A_1 \cos\frac{\pi x}{a}$$

$$\kappa_y = \psi_{y,y} = 0$$

$$\kappa_{xy} = \psi_{x,y} + \psi_{y,x} = \frac{\pi}{a} A_2 \cos\frac{\pi x}{a} \tag{1}$$

$$\epsilon_x^0 = \alpha + \xi \cos\frac{\pi x}{a}$$

$$\epsilon_y^0 = \beta_0 + \zeta \cos\frac{\pi x}{a}$$

$$\gamma_{xy}^0 = \gamma_0 + \eta \cos\frac{\pi x}{a}$$

$$\gamma_{xz} = \psi_x + w_{,x} = \left(A_1 - \frac{\pi}{a} A_3\right) \sin\frac{\pi x}{a}$$

$$\gamma_{yz} = \psi_y + w_{,y} = A_2 \sin\frac{\pi x}{a}$$

where $2a$ is the length of the laminate and $A_1, A_2, A_3, \alpha, \beta_0, \gamma_0, \xi, \zeta$, and η are kinematic constants to be determined. It is noticed that the extensional and shearing membrane strains ϵ_x^0, ϵ_y^0, and γ_{xy}^0 contain sinusodial terms that are in phase with out-of-plane transverse deflection w. However, the two rotations along the x and y directions ψ_x and ψ_y contain sinusodial terms that are 90 deg out of phase with out-of-plane deflection. To satisfy the equations of equilibrium, ψ_y cannot be set to zero trivially. This is different from the case of cylindrical buckling of laminated plates based on classical lamination theory.[5] The compatibility condition

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{2}$$

yields $\zeta = 0$, i.e., the transverse membrane strain $\epsilon_y^0 = \beta_0$ must be a constant. Other compatibility conditions are automatically satisfied. The constitutive relations for the laminates are characterized by the following equations:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{3}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = k \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{4}$$

For simplicity, a single shear correction factor is chosen in this study. However, for a multilayer laminate composed of anisotropic plies, the transverse shear effect generally cannot be characterized by a single number independent of the anisotropic moduli and the stacking sequence.[12] Denoting the thickness of the laminate to be h, the interlaminar stiffnesses are defined as

$$A_{ij} = \int_{-h/2}^{h/2} C_{ij}\, dz, \qquad i,j = 4,5 \tag{5}$$

Based on shear deformation theory, the equations of equilibrium for the one-dimensional buckling problem are

$$N_{x,x} = 0 \tag{6a}$$

$$N_{xy,x} = 0 \tag{6b}$$

$$M_{x,x} - Q_x = 0 \tag{6c}$$

$$M_{xy,x} - Q_y = 0 \tag{6d}$$

$$Q_{x,x} + N_x w_{,xx} = 0 \tag{6e}$$

Substituting Eqs. (3) and (4) into Eq. (6) and using the kinematic relations, Eq. (1), we obtain

$$\begin{bmatrix} A_{11} & A_{16} & B_{11} & B_{16} & 0 \\ A_{16} & A_{66} & B_{16} & B_{66} & 0 \\ B_{11} & B_{16} & D_{11}+\left(\frac{a}{\pi}\right)^2 kA_{55} & D_{16}+\left(\frac{a}{\pi}\right)^2 kA_{55} & \left(\frac{a}{\pi}\right)^2 kA_{55} \\ B_{16} & B_{66} & D_{16}+\left(\frac{a}{\pi}\right)^2 kA_{45} & D_{66}+\left(\frac{a}{\pi}\right)^2 kA_{44} & \left(\frac{a}{\pi}\right)^2 kA_{45} \\ 0 & 0 & \left(\frac{a}{\pi}\right)^2 kA_{55} & \left(\frac{a}{\pi}\right)^2 kA_{45} & \left(\frac{a}{\pi}\right)^2 kA_{45} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \frac{\pi}{a} A_1 \\ \frac{\pi}{a} A_2 \\ -\left(\frac{\pi}{a}\right)^2 A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ N_x A_3 \end{bmatrix} \tag{7}$$

From Eq. (7), the critical buckling load $P_{cr} \equiv -(N_x)_{cr}$ can be determined as

$$P_{cr} = \frac{\pi^2 D}{a^2} \tag{8}$$

where D is referred to as "equivalent bending rigidity" and defined by

$$D \equiv \frac{\det [H]}{\det [H_{55}]} \tag{9}$$

where $[H]$ is the 5×5 matrix on the left-hand side of Eq. (7) and $[H_{55}]$ is the submatrix of $[H]$ formed by omitting its fifth row and fifth column. Because of the unsymmetric layup and transverse shear effect, the equivalent bending rigidity D and the buckling load P_{cr} depend not only on the bending stiffnesses D_{11}, D_{16}, and D_{66} but also on the extensional stiffnesses A_{11}, A_{16}, and A_{66}, the bending-extensional coupling stiffnesses B_{11}, B_{16}, and B_{66}, and the inter-laminar stiffnesses A_{44}, A_{45}, and A_{55}. If the transverse shear effect is neglected, i.e., based on CLT, the critical buckling load of an unsymmetric laminate becomes[5]

$$(P_{cr})_{CLT} = \frac{\pi^2 D_c}{a^2} \tag{10}$$

where

$$D_c = \begin{vmatrix} A_{11} & A_{16} & B_{11} \\ A_{16} & A_{66} & B_{16} \\ A_{11} & A_{16} & B_{11} \end{vmatrix} \begin{vmatrix} A_{11} & A_{16} \\ A_{16} & A_{66} \end{vmatrix}^{-1} \tag{11}$$

is the corresponding equivalent bending rigidity. Comparing with Eq. (9), the stiffness coefficients D_{16}, D_{66}, B_{66}, A_{44}, A_{55}, and A_{45} are absent in Eq. (11). For a symmetric laminate with $B_{ij} = 0$, the critical buckling load is simplified further to

$$(P_{cr})_{CLT, B_{ij}=0} = \frac{\pi^2 D_{11}}{a^2} \tag{12}$$

It can be shown that $D < D_c < D_{11}$. This indicates that both effects of bending-extensional coupling and transverse shear deformation will reduce the equivalent bending rigidity and the critical buckling load.

The quantitative effects of bending-extensional coupling and transverse shear deformation on the buckling strength of a laminated plate can be examined by using antisymmetric angle-ply laminates $[\pm\theta]_n$. The ply mechanical properties of graphite/epoxy composite material are given as follows: $E_1 = 137.9$ GPa, $E_2 = 9.65$ GPa, $G_{12} = G_{13} = 5.52$ GPa, $G_{23} = 4.14$ GPa, and $\nu_{12} = 0.3$. The shear correction factor k is taken as $\pi^2/12$ (Ref. 13). The expression of the ratio D/D_{11}, which is identical to $P_{cr}/(P_{cr})_{CLT, B_{ij}=0}$, for antisymmetric laminates $[\pm\theta]_n$ is given in the Appendix [see Eq. (A5)]. This ratio is calculated for laminate stacking sequence $[\pm\theta]_n$ with n varying from 1 to 5 and for various slenderness ratios L/h ($L = 2a$). The results are shown in Figs. 1–4. Figure 1 shows the relation of the ratio D/D_{11} and the orientation θ based on CLT. It is found that the amount of reduction in equivalent bending rigidity D because of the bending-extensional coupling exceeds 50% for $n = 1$ and 25 deg $< \theta <$ 45 deg. Results based on SDT are given in Figs. 2–4 for three slenderness ratios $L/h = 10$, 20, and 40. Because of effects of both bending-extensional coupling and transverse shear deformation, the maximum reduction in D is about 65% for $n = 1$, $L/h = 10$, and 20 deg $< \theta <$ 25 deg. The amount of reduction decreases as the number of sets n and the slenderness ratio L/h increase. This is because the coupling stiffnesses B_{11}, B_{16}, and B_{66} decrease as n increases, and the transverse shear effect decreases as the slenderness ratio L/h increases. The dotted curves in Figs. 1–4 represent the results without bending-extension coupling effect and give least reduction in D. Figures 2–4 also show that the ratios of D/D_{11} for laminates with smaller θ are smaller than those with larger θ. This is because the transverse shear effect is more significant for laminates with smaller θ. This may be seen from Eq. (A5). Figure 5 shows the effect of transverse shear deformation on normalized equivalent bending rigidity D/D_{11} with different orientations θ by comparing the results from shear deformation theory with those from classical lamination theory.

The postbuckling solution can be obtained by the following procedure. First, the constants A_1, A_2, ξ, and η can be solved

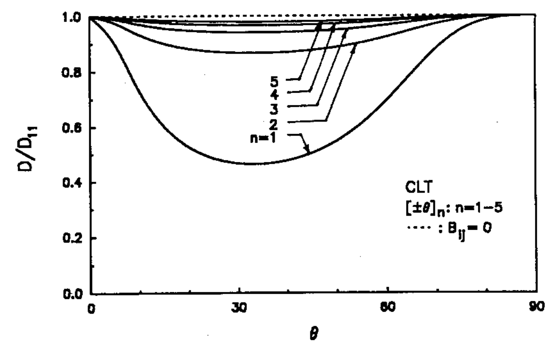

Fig. 1 Equivalent bending rigidity from CLT.

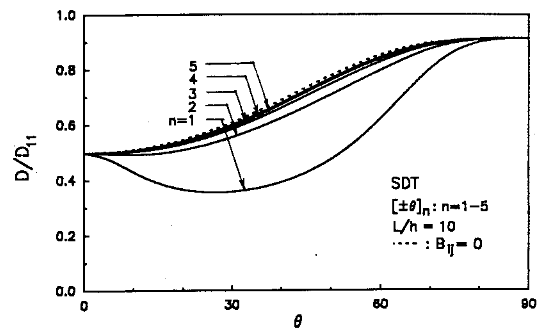

Fig. 2 Equivalent bending rigidity for $L/h = 10$.

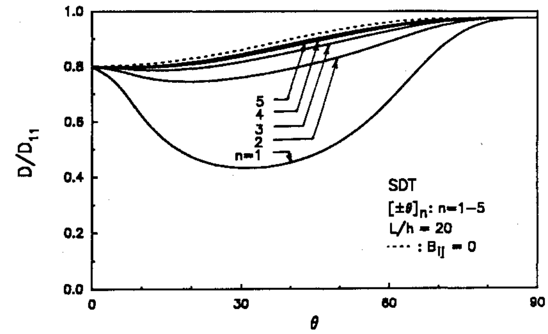

Fig. 3 Equivalent bending rigidity for $L/h=20$.

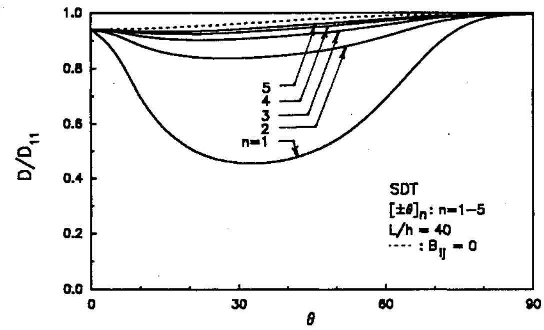

Fig. 4 Equivalent bending rigidity for $L/h=40$.

in terms of A_3 from Eq. (7)

$$\xi=-\frac{\det\,[H_{51}]}{\det\,[H_{55}]}\frac{\pi^2}{a^2}A_3 \tag{13a}$$

$$\eta=\frac{\det\,[H_{52}]}{\det\,[H_{55}]}\frac{\pi^2}{a^2}A_3 \tag{13b}$$

$$A_1=-\frac{\det\,[H_{53}]}{\det\,[H_{55}]}\frac{\pi}{a}A_3 \tag{13c}$$

$$A_2=\frac{\det\,[H_{54}]}{\det\,[H_{55}]}\frac{\pi}{a}A_3 \tag{13d}$$

From Eqs. (1), (3) and (7), one obtains

$$-P\equiv N_x=A_{11}\alpha+A_{12}\beta_0+A_{16}\gamma_0 \tag{14}$$

$$S\equiv N_{xy}=A_{16}\alpha+A_{26}\beta_0+A_{66}\gamma_0 \tag{15}$$

$$N_y=A_{12}\alpha+A_{22}\beta_0+A_{26}\gamma_0+\left[A_{12}\xi+A_{26}\eta_0+B_{12}\left(\frac{\pi}{a}\right)A_1+B_{26}\left(\frac{\pi}{a}\right)A_2\right]\cos\frac{\pi x}{a} \tag{16}$$

It is seen that both the compressive and shearing membrane forces, P and S, are constants. However, Eq. (16) shows that the transverse membrane force N_y is not a constant in general. From Eqs. (14) and (15), the constants α and γ_0 can be solved if the transverse membrane strain β_0 and the in-plane shear force resultant S are given.

Finally, the amplitude A_3 can be determined by the kinematic relation of the total axial shortening $\Delta=2a\epsilon_0$, where ϵ_0 is the compressive strain applied on the edges $x=\pm a$

$$-\int_0^a \epsilon_x^0\,\mathrm{d}x+½\int_0^a w_{,x}^2\,\mathrm{d}x=a\epsilon_0 \tag{17}$$

Thin-Film Strip Delamination

The results from the preceding section are now applied to the thin-film strip delamination with length $2a$ in a thick base laminate with clamped edges. The model is shown in Fig. 6. The base laminate is subjected to constant membrane strain loads

$$\epsilon_x=-\epsilon_0$$
$$\epsilon_y=\beta_0 \tag{18}$$
$$\gamma_{xy}=\gamma_0$$

When these membrane strains are sufficiently large, the delaminated layer buckles and deflects away from the base laminate. It is assumed that the stiffness of the delaminated layer described in Eq. (3) is negligibly small compared with that of the base laminate such that the state of membrane strain in the base laminate is not affected by the buckling deformation of the delaminated layer. Since there is no out-of-plane transverse deflection in the base laminate, the deflection and rotation of the delaminated layer are zero at the two interfaces with the base laminate.

The solution given in the preceding section can be taken as the postbuckling solution of the thin-film strip delamination problem. The postbuckling deformation of the delaminated layer is given in Eq. (1), where $\zeta=0$ and β_0 and γ_0 are the specified membrane strains ϵ_y and γ_{xy} in the base laminate, respectively. The constant α is determined from Eq. (14)

$$\alpha=-\frac{1}{A_{11}}(P_{cr}+A_{12}\beta_0+A_{16}\gamma_0) \tag{19}$$

Using Eqs. (1), (17), and (19), the amplitude A_3 of the transverse deflection $w(x)$ is given as

$$A_3=\frac{2a}{\pi}\left[\epsilon^*-\left(\frac{\pi}{a}\right)^2\frac{D}{A_{11}}\right]^{½} \tag{20}$$

where

$$\epsilon^*=\epsilon_0-\frac{A_{12}\beta_0+A_{16}\gamma_0}{A_{11}} \tag{21}$$

is referred as the "equivalent base-laminate strain"[5] and is a linear combination of the three membrane strains applied on the base laminate. The critical value of ϵ_x^* at the onset of delamination buckling can be decided by setting A_3 to zero in Eq. (20) and yields

$$\epsilon_{cr}^*=\left(\frac{\pi}{a}\right)^2\frac{D}{A_{11}} \tag{22}$$

Similar forms of critical strain for both symmetric and unsymmetric laminates without considering the transverse shear ef-

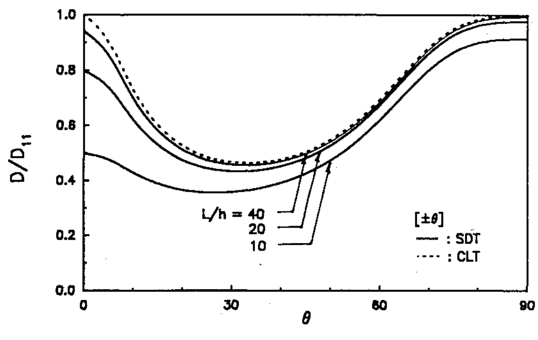

Fig. 5 Transverse shear effect on equivalent bending rigidity.

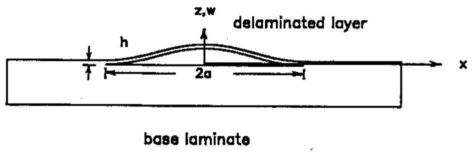

Fig. 6 Thin-film strip delamination model.

fect can be obtained if D in Eq. (22) is replaced by D_{11} and D_c, respectively. Since $D < D_c < D_{11}$, it is indicated that both effects of transverse shear deformation and bending-extension coupling will reduce the critical strain ϵ^*. From Eq. (1), it is found that the maximum deflection $w_{max} = w(0) = 2A_3$, therefore, both effects of transverse shear and bending-extension coupling will increase the maximum transverse deflection as shown in Eq. (20).

Energy Release Rate and Delamination Growth

The energy release rate for the homogeneous orthotropic plate has been obtained by the variational energy principle.[7,10] The same approach is used in the present work for the general unsymmetric laminate. The total potential energy Π for the delaminated layer can be expressed as

$$\Pi = (U_p - U_0) + E_f \tag{23}$$

where U_0 is the strain energy of the delaminated layer in the reference state with the stresses and strains the same as those in the base laminate, and E_f is the fracture energy of the material. The definitions of these terms are expressed as follows:

$$U_p = \tfrac{1}{2}\int_V (\sigma_x\epsilon_x + \sigma_y\epsilon_y + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})\,\mathrm{d}V \tag{24}$$

$$U_0 = \tfrac{1}{2}\int_V (\sigma_{x0}\epsilon_{x0} + \sigma_{y0}\epsilon_{y0} + \tau_{xy0}\gamma_{xy0})\,\mathrm{d}V \tag{25}$$

$$E_f = 2G^*a \tag{26}$$

The variational principle requires

$$\delta\Pi(w, \psi_x, \psi_y, a) = 0 \tag{27}$$

With the delamination length a taken as one of the variational variables, the local growth condition of delamination is the equation corresponding to variational term δa. After a series of routine but tedious variational operations, the analytic form of energy release rate is given as

$$\begin{aligned} G = {} & \tfrac{1}{2}N_x[\epsilon_x^0(a) + \epsilon_0] + \tfrac{1}{2}N_{xy}[\gamma_{xy}^0(a) - \gamma_0] \\ & + \tfrac{1}{2}M_x(a)\kappa_x(a) + \tfrac{1}{2}M_{xy}(a)\kappa_{xy}(a) + \tfrac{1}{2}\epsilon_0(N_x - N_x^0) \\ & - \tfrac{1}{2}\beta_0[N_y(a) - N_y^0] - \tfrac{1}{2}\gamma_0(N_{xy} - N_{xy}^0) \end{aligned} \tag{28}$$

where

$$\begin{aligned} N_x^0 &= A_{11}\epsilon_0 + A_{12}\beta_0 + A_{16}\gamma_0 \\ N_y^0 &= A_{12}\epsilon_0 + A_{22}\beta_0 + A_{26}\gamma_0 \\ N_{xy}^0 &= A_{16}\epsilon_0 + A_{26}\beta_0 + A_{66}\gamma_0 \end{aligned} \tag{29}$$

Using Eqs. (1), (3), (13–16), (19–21), and (29), Eq. (28) can be written as

$$\begin{aligned} G = {} & \frac{A_{11}}{2}\left(\epsilon^* - \frac{\pi^2 D}{a^2 A_{11}}\right)\left\{\epsilon^* - \frac{\pi^2 D}{a^2 A_{11}}\left(1 + 4\,\frac{\det[H_{53}]}{\det[H_{55}]}\right)\right. \\ & \left. -4\,\frac{A_{44}}{A_{11}}\frac{\det[H_{54}]^2}{\det[H_{55}]^2} + 4\,\frac{A_{45}}{A_{11}}\frac{\det[H_{54}]}{\det[H_{55}]}\left(1 + \frac{\det[H_{53}]}{\det[H_{55}]}\right)\right\} \end{aligned} \tag{30}$$

Equation (30) can also be derived by the path-independent J integral method. If the transverse shear effect is neglected, Eq. (30) can be reduced to

$$G = \frac{A_{11}}{2}\left(\epsilon^* - \frac{\pi^2}{a^2}\frac{D_c}{A_{11}}\right)\left(\epsilon^* + 3\,\frac{\pi^2}{a^2}\frac{D_c}{A_{11}}\right) \tag{31}$$

This is identical to the formula obtained from the large deflection classical lamination theory given by Yin.[5] For antisymmetric angle-ply laminates $[\pm\theta]_n$, the formula for energy release rate is given in the Appendix.

The stability characteristics of delamination growth can be determined by the variation of energy release rate with delamination length under a fixed strain state in the base laminate. For a given load parameter ϵ^*, it is found that the energy release rate G increases from zero with $a = (\pi^2 D/A_{11}\epsilon^*)^{1/2}$ to its maximum value, then decreases slowly. When a approaches infinity, G approaches its limiting value $G_\infty = A_{11}(\epsilon^*)^2/2$ (see Fig. 7, where $\bar{a} \equiv a/h$). If the load parameter ϵ^* is sufficiently large such that the energy release rate G in the postbuckling state exceeds the fracture energy of the material G^*, then delamination growth occurs. For low fracture toughness material with $G^* < G_\infty$, since energy release rate G approaches G_∞

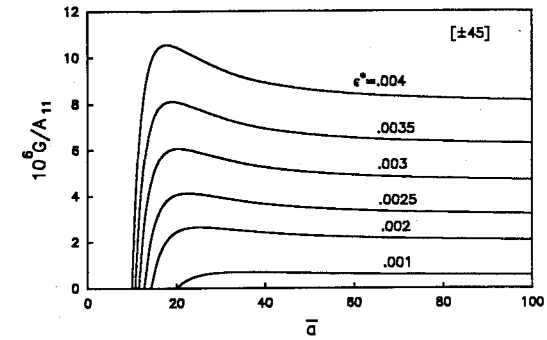

Fig. 7 Variation of energy release rate with delamination length.

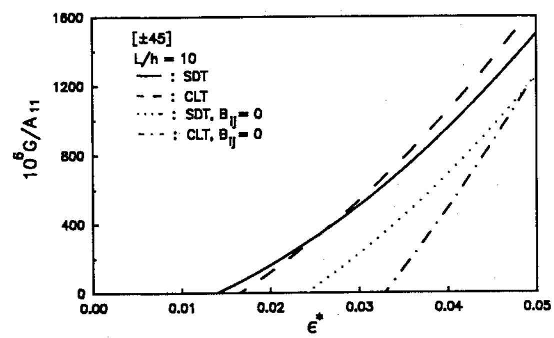

Fig. 8 Comparison of energy release rate between SDT and CLT ($L/h = 10$).

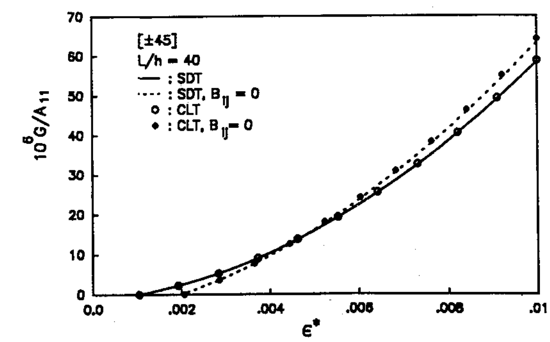

Fig. 9 Comparison of energy release rate between SDT and CLT ($L/h = 40$).

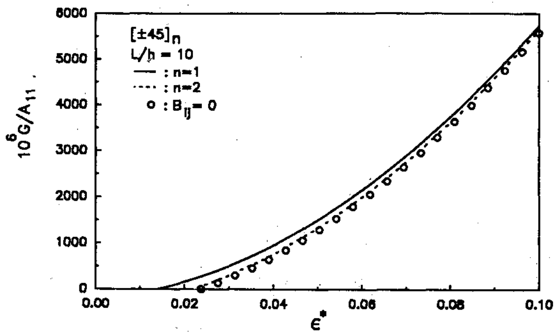

Fig. 10 Coupling effect on energy release rate ($L/h = 10$).

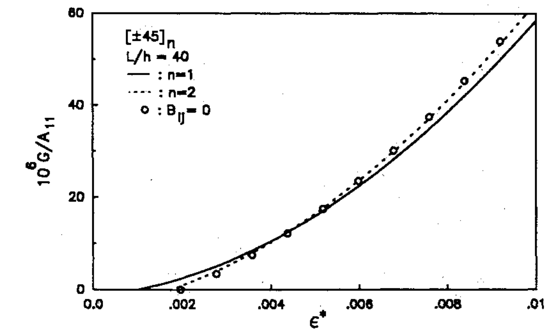

Fig. 11 Coupling effect on energy release rate ($L/h = 40$).

as delamination length a approaches infinity, a catastrophical delamination growth will take place once the delamination length exceeds a specific value such that $G = G^*$. On the other hand, for high fracture toughness material with $G_\infty < G^* < G_{max}$, an unstable delamination growth occurs in the region $G > G^*$, and a final arrested state of delamination growth will be attained when $G = G^*$. Any further delamination growth is stable because it will require a larger load parameter ϵ^*.

Yin[5] has shown that the bending-extension coupling will reduce the slope of the energy release rate with load parameter $dG/d\epsilon^*$. In the present work, it is observed that the transverse shear has the same effect as shown in Eq. (A10). Since the delamination buckling load based on SDT is less than that based on CLT, $G_{SDT} > G_{CLT}$ at the initial postbuckling stage. However, because $(dG/d\epsilon^*)_{SDT} < (dG/d\epsilon^*)_{CLT}$, it turns out that $G_{SDT} < G_{CLT}$ when the load parameter is sufficiently large. Figures 8 and 9 show the variations of normalized energy release rate G/A_{11} with the load parameter ϵ^* for slenderness ratios $L/h = 10$ and 40, respectively. Figures 10 and 11 show the coupling effect on energy release rate for laminates with different ranks of coupling. It is seen that the effect is not significant for number of set $n = 2$.

Conclusions

Based on the current work, the following conclusions are made.

1) The closed-form solutions of one-dimensional buckling for general unsymmetric laminates have been obtained based on large deflection shear deformation theory. It is found that the rotation along the y direction ψ_y is not trivially zero to satisfy the equilibrium equations.

2) The buckling characteristic can be determined by the equivalent bending rigidity D defined in Eq. (9). It is found that $D < D_c < D_{11}$, where D_c is the equivalent bending rigidity for unsymmetric laminates based on classical lamination theory.

3) Both bending-extension coupling and transverse shear deformation will reduce the critical buckling load and increase the postbuckling deformation.

4) The closed-form of the energy release rate can be obtained by a variational energy principle. Both bending-extension coupling and transverse shear deformation will reduce the slope of the energy release rate with load parameter.

Appendix

The postbuckling solutions and the energy release rate for antisymmetric angle-ply laminates $[\pm\theta]_n$ are given in this Appendix.

For antisymmetric angle-ply laminates $[\pm\theta]_n$, the stiffness matrices[14] are

$$[A] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \tag{A1}$$

$$[D] = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \tag{A2}$$

$$[B] = \begin{bmatrix} 0 & 0 & B_{16} \\ 0 & 0 & B_{26} \\ B_{16} & B_{26} & 0 \end{bmatrix} \tag{A3}$$

and matrix $[H]$ defined in Eq. (7) becomes

$$[H] = \begin{pmatrix} A_{11} & 0 & 0 & B_{16} & 0 \\ 0 & A_{66} & B_{16} & 0 & 0 \\ 0 & B_{16} & D_{11} + \dfrac{a^2}{\pi^2} kA_{55} & 0 & \dfrac{a^2}{\pi^2} kA_{55} \\ B_{16} & 0 & 0 & D_{66} + \dfrac{a^2}{\pi^2} kA_{44} & 0 \\ 0 & 0 & \dfrac{a^2}{\pi^2} kA_{55} & 0 & \dfrac{a^2 kA_{55}}{\pi^2} \end{pmatrix} \tag{A4}$$

The ratio of equivalent bending rigidity to bending stiffness D_{11} is

$$\frac{D}{D_{11}} = \left[\frac{1}{1 - (B_{16}^2/A_{66}D_{11})} + \frac{D_{11}}{(a^2/\pi^2)kA_{55}} \right]^{-1} \tag{A5}$$

The kinematic constants ξ, η, A_1, A_2, A_3, and α are

$$\xi = 0 \tag{A6a}$$

$$\eta = -\frac{\pi^2}{a^2} A_3 \frac{(\pi^2/a^2)kA_{55}B_{16}}{A_{66}[(D_{11} + (\pi^2/a^2)kA_{55}] - B_{16}^2} \tag{A6b}$$

$$A_1 = \frac{\pi}{a} A_3 \frac{(\pi^2/a^2)kA_{55}A_{66}}{A_{66}[D_{11} + (\pi^2/a^2)kA_{55}] - B_{16}^2} \tag{A6c}$$

$$A_2 = 0 \tag{A6d}$$

$$A_3 = \frac{2a}{\pi} \left[\epsilon^* - \left(\frac{\pi^2}{a}\right) \frac{D}{A_{11}} \right]^{1/2} \tag{A6e}$$

$$\alpha = -\frac{1}{A_{11}} \left(\frac{\pi^2 D}{a^2} + A_{12}\beta_0 + A_{16}\gamma_0 \right) \tag{A6f}$$

From the previous results, it is seen that, for this special family of laminates $[\pm\theta]_n$, $\epsilon_x^0 = \alpha = \text{const}$ and $\psi_y = \kappa_y = \gamma_{yz} = 0$. The energy release rate for these laminates can be expressed as

$$G = \frac{A_{11}}{2} \left[\epsilon^* - \left(\frac{\pi}{a}\right)^2 \frac{D}{A_{11}} \right] \left\{ \epsilon^* + \left(\frac{\pi}{a}\right)^2 \frac{D}{A_{11}} \right.$$

$$\times\left[3-4\frac{1-(B_{16}^2/A_{66}D_{11})}{(1-B_{16}^2/A_{66}D_{11})+a^2kA_{55}/\pi^2D_{11}}\right]\Bigg\} \tag{A7}$$

If the transverse shear effect is neglected, Eq. (A7) reduces to Eq. (26). If the bending-extension coupling is neglected (i.e., $B_{16}=0$), Eq. (A7) becomes

$$G=\frac{A_{11}}{2}\left(\epsilon^*-\frac{\pi^2D}{a^2A_{11}}\right)\Bigg\{\epsilon^*+\frac{\pi^2D}{a^2A_{11}}$$

$$\times\left[3-\frac{4}{1+(a^2kA_{55}/\pi^2D_{11})}\right]\Bigg\} \tag{A8}$$

where

$$D=\frac{D_{11}}{1+(\pi^2D_{11}/a^2kA_{55})} \tag{A9}$$

The rate of change in the energy release rate with the equivalent base-laminate strain ϵ^* is given by

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon^*}\left(\frac{G}{A_{11}}\right)=\epsilon^*+\frac{\pi^2D}{a^2A_{11}}$$

$$\times\left[1-2\frac{1-(B_{16}^2/A_{66}D_{11})}{1-(B_{16}^2/A_{66}D_{11})+a^2kA_{55}/\pi^2D_{11}}\right] \tag{A10}$$

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